International Journal of Theoretical Physics, Vol. 41, No. 10, October 2002 (© 2002)

Unified Method for Dynamical Groups of Some Anharmonic Potentials

Shi-Hai Dong*

Received February 13, 2002

Realizations of the creation and annihilation operators for some important anharmonic potentials, such as the Morse potential, the modified Pöschl–Teller potential (MPT), the pseudoharmonic oscillator, and infinitely deep square-well potential, are presented by a factorization method. It is shown that the operators for the Morse potential and the MPT potential satisfy the commutation relations of an SU(2) algebra, but those of the pseudoharmonic oscillator and the infinitely deep square-well potential constitute an SU(1, 1) algebra. The matrix elements of some related operators are analytically obtained. The harmonic limits of the SU(2) operators for the Morse and MPT potentials are studied as the Weyl algebra.

KEY WORDS: Algebraic method; anharmonic potential; factorization method.

1. INTRODUCTION

During the past several decades, the algebraic method has been applied to a wide variety of fields in both physics and chemistry. Systems displaying a dynamical symmetry can be solved with algebraic techniques (Arima and Iachello, 1974; Frank and Van Isacker, 1994; Iachello and Levine, 1995). In particular, the Morse (Morse, 1929) and Pöschl–Teller (PT) potentials [Pöschl and Teller, 1933] represent two of the most studied anharmonic systems where these techniques have been used. Both of them are closely related with SO(2, 1) and SU(2) groups (Alhassid *et al.*, 1983; Berrondo and Palma, 1980; Cooper, 1993; Englefield and Quesne, 1991; Frank and Wolf, 1984; Wu and Alhassid, 1990). The latter has been used to describe the vibrational excitations of molecular systems, while the former is associated to the potential group approach. The relation between the SU(2) group and the Morse and PT systems can be directly established by means of a coordinate transformation applied to the radial equation of a 2D harmonic oscillator. Because of its importance in the field of the molecular physics (Child and Halonen, 1984; Jensen, 2000; Nieto and Simmons, 1979), the other different approaches to

[1] Instituto de Ciencias Nucleares, UNAM, A. P. 70-543, Circuito Exterior, C. U., 04510 Mexico, D. F., Mexico; e-mail: dongsh2@yahoo.com.

0020-7748/02/1000-1991/0

study the Morse potential such as the supersymmetry transformation (Benedict and Molnar, 1999), the time-dependent generalizations (Bessis and Bessis, 1994; Kondo and Truax, 1988), the coherent states (Gerry, 1986; Kais and Levine, 1990) as well as the path integration method (Berceanu and Gheorghe, 1987) have also been carried out. The PT potential has been applied in the framework of the SU(2) vibron model, where it is associated to the vibrational excitations of the molecular bending modes (Iachello and Oss, 1993). In addition, the pseudoharmonic oscillator, as another important molecular potential, has couched for a while and is recognized gradually (Ballhausen, 1998; Büyükkilic *et al.*, 1992; Goldmen *et al.*, 1960; Popov, 2001). The aim of this work was to establish ladder operators for the respective anharmonic potentials with the factorization method (Infeld and Hull, 1951) and then construct their dynamic groups. It is shown that the operators for the Morse and modified Pöschl–Teller (MPT) potentials satisfy the SU(2) group, but the SU(1, 1) for the pseudoharmonic potential and the infinitely deep square-well potential.

This paper is organized as follows: In Section 2, we establish the ladder operators directly from their eigenfunctions and then constitute their suitable algebras. The matrix elements of the related operators are analytically obtained from the ladder operators. Section 3 is devoted to showing how the harmonic limits of SU(2) algebra for the Morse and MPT potentials are contracted to the Weyl algebra. Conclusions are given in Section 4.

2. CONSTRUCTIONS OF THE LADDER OPERATORS

In this section we address how to find the ladder operators for the wave functions with the factorization method. We intend to find differential operators $\hat{\mathcal{O}}_{\pm}$ with the following property:

$$\hat{\mathcal{O}}_{\pm}\psi_n(\xi) = o_{\pm}\psi_{n\pm 1}(\xi). \tag{1}$$

Specifically, we look for operators of the form

$$\hat{\mathcal{O}}_{\pm} = A_{\pm}(\xi)\frac{d}{d\xi} + B_{\pm}(\xi), \tag{2}$$

where we stress that these operators depend only on the physical variable ξ. The physical variable ξ is different with respect to the different cases, as is shown later. We first study the case of Morse potential.

2.1. Morse Potential

Choosing the separated atoms limit as the zero of energy, the Morse potential has the following form (Morse, 1929):

$$V(x) = V_0(e^{-2\beta x} - 2e^{-\beta x}), \tag{3}$$

where $V_0 > 0$ corresponds to its depth, β is related with the range of the potential, and x gives the relative distance from the equilibrium position of the atoms.

The solution of the Schrödinger equation associated to the Morse potential is given by (Landau and Lifshitz, 1977)

$$|n\rangle_\nu \equiv \psi_n^\nu(y) = N_n^\nu\, e^{-\frac{y}{2}} y^s\, L_n^{2s}(y), \tag{4}$$

where $L_n^{2s}(y)$ are the associated Laguerre functions, the argument y is related with the physical displacement coordinate x by $y = \nu\, e^{-\beta x}$, N_n^ν is the normalization constant

$$N_n^\nu = \sqrt{\frac{\beta(\nu - 2n - 1)\Gamma(n+1)}{\Gamma(\nu - n)}}, \tag{5}$$

and the variables ν and s are related with the potential and the energy, respectively, through

$$\nu = \sqrt{\frac{8\mu V_0}{\beta^2\hbar^2}}, \qquad s = \sqrt{\frac{-2\mu E}{\beta^2\hbar^2}}, \tag{6}$$

with the constraint condition $2s = \nu - 2n - 1$, where μ is the reduced mass of the molecule.

Let us seek the ladder operators for this system. We start by establishing the action of the differential operator $\frac{d}{dy}$ on the Morse functions (4)

$$\frac{d}{dy}|n\rangle_\nu = \left[-\frac{1}{2} + \frac{s}{y}\right]|n\rangle_\nu + N_n^\nu\, e^{-\frac{y}{2}} y^s\, \frac{d}{dy} L_n^{2s}(y). \tag{7}$$

One possible relation for the derivative of the associated Laguerre functions is given by (Gradshteyn and Ryzhik, 1994)

$$\frac{d}{dy} L_n^\alpha(y) = -\frac{1}{(\alpha+1)}\left[y L_{n-1}^{\alpha+2}(y) + n L_n^\alpha(y)\right]. \tag{8}$$

Substitution of this expression into (7) allows us to obtain the following relation between the Morse functions belonging to the same potential

$$\left[\frac{d}{dy}(2s+1) - \left(\frac{1}{y}s - \frac{1}{2}\right)(2s+1) + n\right]|n\rangle_\nu = -\frac{N_n^\nu}{N_{n-1}^\nu}|n-1\rangle_\nu, \tag{9}$$

from which we can define the operator

$$\hat{K}_- = -\left[\frac{d}{dy}(2s+1) - \frac{1}{y}s(2s+1) + \frac{\nu}{2}\right]\sqrt{\frac{s+1}{s}} \tag{10}$$

with the following effect over the wave functions

$$\hat{K}_-|n\rangle_\nu = k_-|n-1\rangle_\nu = \sqrt{n(\nu - n)}|n-1\rangle_\nu. \tag{11}$$

As we can see, this operator annihilates the ground state $|0\rangle_\nu$, as expected from a step-down operator. The variable s in Eq. (10) is understood as a diagonal operator depending on n, according to $2s = \nu - 2n - 1$. Also note that the order of the different terms in (10) is important, as these operators do not commute.

We now proceed to find the corresponding creation operator. We first need to obtain a relation between $\frac{d}{dy}L_n^\alpha(y)$ and $L_{n+1}^{\alpha-2}(y)$, since this implies a relation between $\frac{d}{dy}|n\rangle_\nu$ and the Morse function $|n-1\rangle_\nu$. To this end we start with the relation

$$y\,\frac{d}{dy}L_n^\alpha(y) = nL_n^\alpha(y) - (n+\alpha)L_{n-1}^\alpha(y), \tag{12}$$

which, when taking into account that (Gradshteyn *et al.*, 1994)

$$(n+1)L_{n+1}^\alpha(y) - (2n+\alpha+1-y)L_n^\alpha(y) + (n+\alpha)L_{n-1}^\alpha(y) = 0 \tag{13}$$

can be transformed into

$$y\,\frac{d}{dy}L_n^\alpha(y) = (-n-\alpha-1+y)L_n^\alpha(y) + (n+1)L_{n+1}^\alpha(y). \tag{14}$$

On the other hand, the relation

$$L_n^{\alpha-1}(y) = L_n^\alpha(y) - L_{n-1}^\alpha(y), \tag{15}$$

together with Eq. (13) allows to set up the result

$$\frac{(\alpha-1)}{(n+\alpha)}L_{n+1}^\alpha(y) = \left[\frac{(\alpha+x-1)}{(\alpha+n)}\right]L_n^\alpha(y) + L_{n+1}^{\alpha-2}(y), \tag{16}$$

which in turn can be substituted into Eq. (14) to obtain

$$(\alpha-1)\,\frac{d}{dy}L_n^\alpha(y) = \left[(\alpha+n) - \frac{\alpha(\alpha-1)}{y}\right]L_n^\alpha(y) + \frac{(n+1)(n+\alpha)}{y}L_{n+1}^{\alpha-2}(y). \tag{17}$$

Finally, when this equation is substituted into (7), we obtain

$$\frac{d}{dy}|n\rangle_\nu = \left[-\frac{1}{2} - \frac{s}{y} + \frac{(2s+n)}{(2s-1)}\right]|n\rangle_\nu + \frac{N_n^\nu}{N_{n+1}^\nu}\,\frac{(n+1)(n+2s)}{(2s-1)}|n+1\rangle_\nu, \tag{18}$$

which allows to define the creation operator as

$$\hat{K}_+ = \left[\frac{d}{dy}(2s-1) + \frac{1}{y}s(2s-1) - \frac{\nu}{2}\right]\sqrt{\frac{s-1}{s}} \tag{19}$$

satisfying the equation

$$\hat{K}_+|n\rangle_\nu = k_+|n+1\rangle_\nu = \sqrt{(n+1)(\nu-n-1)}|n+1\rangle_\nu. \tag{20}$$

Since $\hat{K}_+$ is a raising operator, it is expected to annihilate the last bounded state. Indeed, for such a state $s = 1$ and the square root in (19) makes the operator vanish.

We now study the algebra related with operators $\hat{K}_+$ and $\hat{K}_-$. On the basis of the results (11) and (20) we can calculate the commutator $[\hat{K}_+, \hat{K}_-]$:

$$[\hat{K}_+, \hat{K}_-]|n\rangle_\nu = 2k_0|n\rangle_\nu, \tag{21}$$

where we introduce the eigenvalue

$$k_0 = n - \frac{\nu - 1}{2}. \tag{22}$$

We can thus define the operator

$$\hat{K}_0 = \hat{n} - \frac{\nu - 1}{2}. \tag{23}$$

Thus the operators $\hat{K}_\pm$ and $\hat{K}_0$ satisfy the commutation relations

$$[\hat{K}_+, \hat{K}_-] = 2\hat{K}_0, \qquad [\hat{K}_0, \hat{K}_-] = -\hat{K}_-, \qquad [\hat{K}_0, \hat{K}_+] = \hat{K}_+, \tag{24}$$

which correspond to the SU(2) group for the Morse potential. The Casimir operator is

$$\hat{C}|n\rangle_\nu = \left[\hat{K}_0^2 + \frac{1}{2}(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+)\right]|n\rangle_\nu = j(j+1)|n\rangle_\nu, \tag{25}$$

where j, the label of the irreducible representations of SU(2), is given by

$$j = \frac{\nu - 1}{2} = \frac{N}{2}, \tag{26}$$

where we have used the definition $N = \nu - 1$.

From the commutation relations (24) we know that $\hat{K}_0$ is the projection of the angular momentum m, and consequently

$$n - \frac{\nu - 1}{2} = m. \tag{27}$$

Therefore the ground state corresponds to $m = -j$, while the maximum number of states $n_{\max} = (\nu - 3)/2$ and consequently $m_{\max} \mid n_{\max} = -1$. The Morse wave functions are then associated to one branch (in this case to $m \leq -1$) of the SU(2) representations, as expected in Frank and Van Isacker (1994). Finally, we should notice that from the SU(2) algebra the Hamiltonian acquires the simple form

$$\hat{H} = \frac{\hbar\omega}{\nu}\hat{K}_0^2, \tag{28}$$

where

$$\omega = \frac{\hbar\beta^2\nu}{2\mu}. \tag{29}$$

For the wave functions

$$|n\rangle_\nu = \mathcal{N}_n^\nu \hat{K}_+^n |0\rangle_\nu, \tag{30}$$

where the normalization constant is obtained through the commutation relations (24), and turns out to be

$$\mathcal{N}_n^{\nu} = \sqrt{\frac{(\nu - n - 1)!}{n!(\nu - 1)!}}. \tag{31}$$

The following expressions from the operators $\hat{K}_{\pm,0}$ can be obtained as

$$\frac{d}{dy} = \hat{K}_+ \left[\frac{1}{2(2s-1)} \sqrt{\frac{s}{s-1}} \right] - \hat{K}_- \left[\frac{1}{2(2s+1)} \sqrt{\frac{s}{s+1}} \right] + \frac{\nu}{2(2s+1)(2s-1)} \tag{32}$$

and

$$\frac{1}{y} = \hat{K}_+ \left[\frac{1}{2(2s-1)} \sqrt{\frac{s}{s-1}} \right] + \hat{K}_- \left[\frac{1}{2(2s+1)} \sqrt{\frac{s}{s+1}} \right] + \frac{\nu}{2(2s+1)(2s-1)}. \tag{33}$$

The matrix elements of these two functions can be analytically obtained in terms of Eqs. (11) and (20) as

$$\left\langle m \left| \frac{1}{y} \right| n \right\rangle = \frac{1}{(\nu - 2n - 2)} \sqrt{\frac{(n+1)(\nu - n - 1)}{(\nu - 2n - 1)(\nu - 2n - 3)}} \delta_{m,n+1} + \frac{1}{(\nu - 2n)} \sqrt{\frac{n(\nu - n)}{(\nu - 2n - 1)(\nu - 2n + 1)}} \delta_{m,n-1} + \frac{\nu}{(\nu - 2n - 2)(\nu - 2n)} \delta_{m,n} \tag{34}$$

$$\left\langle m \left| \frac{d}{dy} \right| n \right\rangle = \frac{1}{2(\nu - 2n - 2)} \sqrt{\frac{(n+1)(\nu - n - 1)(\nu - 2n - 1)}{(\nu - 2n - 3)}} \delta_{m,n+1} - \frac{1}{2(\nu - 2n)} \sqrt{\frac{n(\nu - n)(\nu - 2n - 1)}{(\nu - 2n + 1)}} \delta_{m,n-1} + \frac{\nu}{2(\nu - 2n)(\nu - 2n - 2)} \delta_{m,n} \tag{35}$$

2.2. MPT Potential

We start by presenting the eigenfunctions for the MPT problem (Landau and Lifshitz, 1977). The MPT potential as described in Flügge (1971) can be written as

$$V(x) = -\frac{D}{\cosh^2(\alpha x)}, \tag{36}$$

where D is the depth of the well and α is related with the range of the potential, while x gives the relative distance from the equilibrium position. The Schrödinger equation associated to this potential is given by

$$\frac{d^2}{dx^2}\Psi_n^2(x) + \frac{2\mu}{\hbar^2}\left(E + \frac{D}{\cosh^2(\alpha x)}\right)\Psi_n^q(x) = 0, \tag{37}$$

where μ is the reduced mass of the molecule and q is related with the depth of the potential as is shown later. We now introduce the following variables in accordance with Landau and Lifshitz (1977),

$$\epsilon = \sqrt{\frac{-2\mu E}{\alpha^2\hbar^2}}, \qquad q(q+1) = \frac{2\mu D}{\alpha^2\hbar^2}, \qquad q = \frac{1}{2}(-1+2k), \tag{38}$$

with

$$k = \sqrt{\frac{1}{4} + \frac{2\mu D}{\alpha^2\hbar^2}}, \qquad \nu = 2k = 2q+1, \tag{39}$$

where ν has been introduced because of its relevance for the identification of the ladder operators with the SU(2) algebra (as shown in the next section). In terms of the variable $u = \tanh(\alpha x)$, the solutions of Eq. (37) are given by

$$|n\rangle_q \equiv \Psi_n^q(u) = (1-u^2)^{\epsilon/2} F\left[\epsilon - q, \epsilon + q + 1, \epsilon + 1; \frac{1}{2}(1-u)\right]. \tag{40}$$

where $F[\epsilon - q, \epsilon + q + 1, \epsilon + 1; \frac{1}{2}(1-u)]$ are hypergeometric polynomials of degree n with the constraint $\epsilon - q = -n$, where $n = 0, 1, 2, \ldots$, for $|n\rangle_q$ to remain finite for $u = -1$. The eigenvalue can be determined by the condition $q - \epsilon = n$ and expressed as

$$E_n = -\frac{\alpha^2\hbar^2}{2\mu}(q-n)^2, \tag{41}$$

where $\epsilon = q - n > 0$. The number of bound states is determined by the dissociation limit $\epsilon = q - n = 0$. The normalization constant, however, was not given in Eq. (40) and must be determined. As we know, the relation between the Gegenbauer polynomials and the hypergeometric functions (Wang and Guo, 1989) can be written as

$$C_n^{\lambda}(x) = \frac{\Gamma(2\lambda + n)}{n!\Gamma(2\lambda)} F\left[-n, 2\lambda + n, \frac{1}{2} + \lambda; \frac{1-x}{2}\right]. \tag{42}$$

Substitution of this expression into Eq. (40) allows to write the following solutions

$$|n\rangle_q = N_n^q (1-u^2)^{\frac{\epsilon}{2}} C_n^{q+\frac{1}{2}-n}(u), \tag{43}$$

where N_n^q is the normalization constant to be determined. To achieve this task we shall consider the following expression (Gradshteyn and Ryzhik, 1994)

$$\int_{-1}^{1} (1-x)^{\nu-3/2}(1+x)^{\nu-\frac{1}{2}}\left[C_n^{\nu}(x)\right]^2 dx = \frac{\pi^{\frac{1}{2}}\Gamma\left(\nu - \frac{1}{2}\Gamma\right)(2\nu+n)}{n!\Gamma(\nu)\Gamma(2\nu)} \times \left[\mathrm{Re}\,\nu > \frac{1}{2}\right], \tag{44}$$

from which we find the result

$$\begin{aligned}\int_{-1}^{1}(1-x^2)^{\nu-3/2}(1+x)\left[C_n^{\nu}(x)\right]^2 dx &= \int_{-1}^{1}(1-x^2)^{\nu-3/2}\left[C_n^{\nu}(x)\right]^2 dx \\ &\quad + \int_{-1}^{1}(1-x^2)^{\nu-3/2}x\left[C_n^{\nu}(x)\right]^2 dx, \\ &= \frac{\pi^{\frac{1}{2}}\Gamma\left(\nu-\frac{1}{2}\right)\Gamma(2\nu+n)}{n!\Gamma(\nu)\Gamma(2\nu)}\end{aligned} \tag{45}$$

as a consequence of the odd parity of the function $(1-x^2)^{\nu-3/2}x[C_n^{\nu}(x)]^2$, whose integral vanishes in the interval $[-1, 1]$.

The normalization is then given by condition

$$\langle n \mid n\rangle_q = \frac{\left(N_n^q\right)^2}{\alpha}\int_{-1}^{1}(1-u^2)^{q-n-1}\left[C_n^{q-n+\frac{1}{2}}(u)\right]^2 du = 1, \tag{46}$$

which leads to the normalization constant

$$N_n^q = \sqrt{\frac{\alpha n!\left(q-n-\frac{1}{2}\right)!(2q-2n)!}{\pi^{\frac{1}{2}}(q-n-1)!(2q-n)!}} \tag{47}$$

once Eq. (45) is taken into account. We should note that for q integer the state associated with null energy is not normalizable. In this case the last bounded state corresponds to $q - n = 1$. We thus have that $n_{\max} = q - 1 = (\nu - 3)/2$.

We now address the problem of finding ladder operators with the factorization method. The ladder operators can be obtained by acting of the differential operator $\frac{d}{du}$ on the MPT wave functions. Therefore, formula (Gradshteyn and

Ryzshik, 1994)

$$\frac{dC_n^{\lambda}(t)}{dt} = 2\lambda C_{n-1}^{\lambda+1}(t), \tag{48}$$

together with Eq. (43), allows to obtain

$$\frac{d}{du}|n\rangle_q = -\frac{u(q-n)}{1-u^2}|n\rangle_q + \frac{2q-2n+1}{\sqrt{1-u^2}}\frac{N_n^q}{N_{n-1}^q}|n-1\rangle_q, \tag{49}$$

and introducing the explicit form of the normalization constant, Eq. (49) becomes

$$\sqrt{1-u^2}\left(\frac{d}{du} + \frac{u(q-n)}{1-u^2}\right)\sqrt{\frac{q-n+1}{q-n}}\Psi_n^q(u) = \sqrt{n(2q-n+1)}\Psi_{n-1}^q(u), \tag{50}$$

from which we can define the annihilation operator $\hat{P}_-$ as

$$\hat{P}_- = \sqrt{1-u^2}\left(\frac{d}{du} + \frac{u(q-n)}{1-u^2}\right)\sqrt{\frac{q-n+1}{q-n}}, \tag{51}$$

or in terms of ν defined in (39)

$$\hat{P}_- = \sqrt{1-u^2}\left(\frac{d}{du} + \frac{u}{1-u^2}\,\epsilon\right)\sqrt{\frac{\epsilon+1}{\epsilon}}, \tag{52}$$

where in order to simplify the notation we have taken into account that $2\epsilon = \nu - 2n - 1 = 2q - 2n$. The action of the operator (52) on the wave functions is then given by

$$\hat{P}_-|n\rangle_q = p_-|n-1\rangle_q = \sqrt{n(\nu-n)}|n-1\rangle_q. \tag{53}$$

As we can see, this operator annihilates the ground state $|0\rangle$, as expected from a lowering operator.

We now proceed to find the corresponding creation operator $\hat{P}_+$. To this end, we consider the formula (Talman, 1968)

$$2(\lambda-1)(2\lambda-1)xC_n^{\lambda}(x) = 4\lambda(\lambda-1)(1-x^2)C_{n-1}^{\lambda+1}(x) + (2\lambda+n-1)(n+1)C_{n+1}^{\lambda+1}(x). \tag{54}$$

This recurrence relation can be used together with Eq. (48) to obtain

$$\frac{d}{du}|n\rangle_q = \frac{u(q-n)}{1-u^2}|n\rangle_q - \frac{(n+1)(2q-n)}{\sqrt{1-u^2}(2q-2n-1)}\frac{N_n^q}{N_{n+1}^q}|n+1\rangle_q. \tag{55}$$

By using the explicit form of the normalization constant (47), Eq. (49) becomes

$$-\sqrt{1-u^2}\left(\frac{d}{du}-\frac{u(q-n)}{1-u^2}\right)\sqrt{\frac{q-n-1}{q-n}}|n\rangle = \sqrt{(n+1)(2q-n)}|n+1\rangle_q. \tag{56}$$

Likewise, in terms of the variable ν, we can thus define the creation operator $\hat{P}_+$ as

$$\hat{P}_+ = \sqrt{1-u^2}\left(-\frac{d}{du}+\frac{u}{1-u^2}\,\epsilon\right)\sqrt{\frac{\epsilon-1}{\epsilon}}, \tag{57}$$

with the following effect on the wave functions

$$\hat{P}_+|n\rangle_q = p_+|n+1\rangle_q = \sqrt{(n+1)(\nu-n-1)}|n+1\rangle_q. \tag{58}$$

Since $\hat{P}_+$ is a raising operator it is expected to annihilate the last bounded state. Indeed, for such state $\epsilon = 1$ and the square root in (57) makes the operator vanish.

We now establish the algebra associated with the operators $\hat{P}_\pm$. On the basis of Eqs. (53) and (58), we calculate the commutator $[\hat{P}_-, \hat{P}_+]$:

$$[\hat{P}_+, \hat{P}_-]|n\rangle_q = 2p_0|n\rangle_q, \tag{59}$$

where we have introduced the eigenvalue

$$p_0 = n - \frac{\nu-1}{2}. \tag{60}$$

We can thus define the operator

$$\hat{P}_0 = \hat{n} - \frac{\nu-1}{2}. \tag{61}$$

The operators $\hat{P}_{\pm,0}$ satisfy the commutation relations

$$[\hat{P}_+, \hat{P}_-] = 2\hat{P}_0, \qquad [\hat{P}_0, \hat{P}_-] = \hat{P}_-, \qquad [\hat{P}_0, \hat{P}_+] = \hat{P}_+, \tag{62}$$

which correspond to the SU(2) algebra. This result is consistent with the description of finite discrete spectrum, in accordance with previous algebraic descriptions of the bounded states of the PT potential (Frank and Van Isacker, 1994). The Casimir operator

$$\hat{C}|n\rangle_q = \left[\hat{P}_0^2 + \frac{1}{2}(\hat{P}_+\hat{P}_- + \hat{P}_-\hat{P}_+)\right]|n\rangle_q = j(j+1)|n\rangle_q, \tag{63}$$

where j, the label of the irreducible representations of the SU(2), is given by

$$j = \frac{\nu-1}{2} = \frac{N}{2}. \tag{64}$$

From the commutation relations (62), we know that $\hat{P}_0$ is the projection of the angular momentum m, and consequently

$$n - \frac{\nu - 1}{2} = m. \tag{65}$$

The ground state thus corresponds to $m = -j$, while the maximum number of quanta $n_{\max} = (\nu - 3)/2$ and consequently $m_{\max} \mid n_{\max} = -1$ in accordance with the constraint condition $\epsilon = q - n = 1$ for the last bounded state. The MPT wave functions are thus associated to one branch (in this case to $m \leq -1$) of the SU(2) representations, as expected. Finally we should notice that in terms of the SU(2) algebra, the Hamiltonian acquires the simple form

$$\hat{H} = \frac{\hbar\omega}{\nu} \hat{P}_0^2, \tag{66}$$

where

$$\omega = \frac{\hbar \beta^2 \nu}{2\mu}.$$

While the wave functions

$$|n\rangle_q = \mathcal{N}_n^\nu \hat{P}_+^n |0\rangle_q, \tag{67}$$

where the normalization constant is obtained through the commutation relations (62), turns out to be

$$\mathcal{N}_n^\nu = \sqrt{\frac{(\nu - n - 1)!}{n!(\nu - 1)!}} \tag{68}$$

For other calculations one can obtain the following expressions in terms of the raising and lowering operators $\hat{P}_\pm$

$$\frac{u}{\sqrt{1 - u^2}} = \frac{1}{2} \left(\hat{P}_- \sqrt{\frac{1}{\epsilon(\epsilon + 1)}} + \hat{P}_+ \sqrt{\frac{1}{\epsilon(\epsilon - 1)}} \right), \tag{69}$$

$$\sqrt{1 - u^2} \frac{d}{du} = \frac{1}{2} \left(\hat{P}_- \sqrt{\frac{1}{(\epsilon + 1)}} - \hat{P}_+ \sqrt{\frac{1}{(\epsilon - 1)}} \right), \tag{70}$$

where it has to be understood that for the last bounded state ($\epsilon = 1$) the raising operator vanishes. On the other hand, we remark that the variable ϵ is to be considered as an n-dependent operator. Using Eqs. (53) and (58) and considering the constraint condition $2\epsilon = \nu - 2n - 1$, we can thus calculate the matrix elements of these functions as

$$\left\langle n' \left| \frac{u}{\sqrt{1 - u^2}} \right| n \right\rangle_q = \langle n' | \sinh(\alpha x) | n \rangle_q$$

$$= \sqrt{\frac{n(\nu-n)}{(\nu-2n-1)(\nu-2n+1)}}\delta_{n',n-1} + \sqrt{\frac{(n+1)(\nu-n-1)}{(\nu-2n-1)(\nu-2n-3)}}\delta_{n',n+1} \quad (71)$$

$$\left\langle n' \left| \sqrt{1-u^2}\frac{d}{du} \right| n \right\rangle_q = \left\langle n' \left| \frac{\cosh(\alpha x)}{\alpha}\frac{d}{dx} \right| n \right\rangle_q = \frac{1}{2}\sqrt{\frac{n(\nu-n)(\nu-2n-1)}{(\nu-2n+1)}}\delta_{n',n-1} - \frac{1}{2}\sqrt{\frac{(n+1)(\nu-n-1)(\nu-2n-1)}{(\nu-2n-3)}}\delta_{n',n+1} \quad (72)$$

2.3. Pseudoharmonic Oscillator

Generally, this potential can be taken as (Goldman *et al.*, 1960),

$$V_{\mathrm{PH}}(r) = \frac{1}{8}\kappa r_0^2\left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2, \quad (73)$$

where κ is the force constant and the r_0 equilibrium bond length. For simplicity, the natural units $\hbar = \mu = \kappa = \omega = 1$ are employed throughout this paper, if not explicitly stated otherwise, where μ is the reduced mass and ω the frequency. Consider the Schrödinger equation with a potential $V(r)$ that depends only on the distance r from the origin

$$H\Psi_{n\ell m}(\theta,\varphi,r) = \left(-\frac{1}{2}\nabla^2 + V_{\mathrm{PH}}(r)\right)\Psi_{n\ell m}(\theta,\varphi,r) = E\Psi_{n\ell m}(\theta,\varphi,r). \quad (74)$$

Let

$$\Psi_{n\ell m}(r,\theta,\varphi) = r^{-1}R_n^{\ell}(r)Y_{\ell m}(\theta,\varphi), \quad (75)$$

where $Y_{\ell m}(\theta,\varphi)$ is the normalized spherical harmonic. Substitution of Eq. (75) into Eq. (74) enables us to obtain the following radial Schrödinger equation

$$\frac{d^2R_n^{\ell}(r)}{2dr^2} + \left[E + V_{\mathrm{PH}}(r) - \frac{\ell(\ell+1)}{2r^2}\right]R_n^{\ell}(r) = 0, \quad (76)$$

where E denotes the energy. If we consider the contribute of effective potential is from the combination of the centrifugal potential with the pseudoharmonic one, we then have

$$V_\ell = \frac{1}{8} r_0^2 \left(\frac{r}{r_0} - \frac{r_0}{r} \right)^2 + \frac{\ell(\ell+1)}{2r^2}, \tag{77}$$

which can be arranged to

$$V_\ell = \frac{1}{8} r_\beta^2 \left(\frac{r}{r_\beta} - \frac{r_\beta}{r} \right)^2 + \frac{1}{4}\left(r_\beta^2 - r_0^2\right), \tag{78}$$

where

$$r_\beta = \sqrt{2}(\beta^2 - 1/4)^{1/4}, \tag{79}$$

with

$$\beta = \sqrt{(\ell + 1/2)^2 + \left(r_0^2/2\right)^2}.$$

The solution of the radial Schrödinger Eq. (76) with the effective potential (78) can be analytically obtained as (Sage, 1984)

$$|n\rangle_\beta = N_n^\beta r^{\beta+1/2} e^{-r^2/4} L_n^\beta (r^2/2), \tag{80}$$

with

$$N_n^\beta = \sqrt{\frac{n!}{2^\beta (n+\beta)!}}. \tag{81}$$

The corresponding eigenvalue can be taken as

$$E = n + \frac{1}{2} + \frac{\beta}{2} - \frac{r_0^2}{4}. \tag{82}$$

We now consider the eigenvalue E under the limit of r_0. When r_0 is very large, the eigenvalue E becomes

$$E \simeq n + \frac{1}{2} + \frac{\ell(\ell+1)}{2r_0^2} + \frac{1}{8r_0^2}, \tag{83}$$

which corresponds to the energy levels of the harmonic oscillator and rigid rotator except for a small constant $1/8r_0^2$. However, for the small r_0, the corresponding eigenvalue E can be taken as

$$E \simeq n + \frac{\ell}{2} + \frac{3}{4}, \tag{84}$$

which is in proportion to the energy levels of the isotropic 3D harmonic oscillator with principle quantum number $2n + \ell$ and force constant $1/4$. In the following section we make use of the radial eigenfunctions (80) to construct the creation and annihilation operators with the factorization method.

Let us address how to find the ladder operators for the pseudoharmonic radial wave functions (80). We start by establishing the action of the differential operator $\frac{d}{dr}$ on the radial wave functions (80)

$$\frac{d}{dr}|n\rangle_\beta = \left[-\frac{r}{2} + \frac{\beta + 1/2}{r}\right]|n\rangle_\beta + N_n^\beta r^{\beta+1/2} e^{-r^2/4} \frac{d}{dr} L_n^\beta(r^2/2). \tag{85}$$

One possible relation for the derivative of the associated Laguerre functions is given in (Gradshteyn and Ryzhik, 1994)

$$x \frac{d}{dx} L_n^\alpha(x) = n L_n^\alpha(x) - (n+\alpha) L_{n-1}^\alpha(x). \tag{86}$$

The substitution of this expression into (85) enables us to obtain the following relation

$$\left(-\frac{d}{dr} + \frac{\beta + 1/2}{r} - \frac{r}{2} + \frac{2n}{r}\right)|n\rangle_\beta = \frac{2(n+\beta)}{r} \frac{N_n^\beta}{N_{n-1}^\beta}|n-1\rangle_\beta. \tag{87}$$

Making use of Eq. (81), we can define the following operator

$$\hat{\mathcal{L}}_- = \frac{1}{2}\left[-r\frac{d}{dr} - \frac{1}{2}r^2 + \left(2n + \beta + \frac{1}{2}\right)\right], \tag{88}$$

with the following effect on the wave function

$$\hat{\mathcal{L}}_-|n\rangle_\beta = \ell_-|n-1\rangle_\beta = \sqrt{n(\beta+n)}|n-1\rangle_\beta. \tag{89}$$

As we can see, this operator annihilates the ground state $|0\rangle_\beta$, as expected from a step-down operator.

We now proceed to find the corresponding creation operator. Before proceeding to do so, we should make use of another relation between the associated Laguerre functions (Gradshteyn and Ryzhik, 1994)

$$x \frac{d}{dx} L_n^\alpha(x) = (n+1) L_{n+1}^\alpha(x) - (n + \alpha + 1 - x) L_n^\alpha(x). \tag{90}$$

Substitution of this expression into Eq. (91) admits us to obtain

$$\left[\frac{d}{dr} - \frac{\beta + 1/2}{r} + \frac{r}{2} + \frac{2}{r}\left(n + \beta + 1 - \frac{r^2}{2}\right)\right]|n\rangle_\beta = \frac{2(n+1)}{r} \frac{N_n^\beta}{N_{n+1}^\beta}|n+1\rangle_\beta. \tag{91}$$

Using Eq. (81) again, we can define the following operator

$$\hat{\mathcal{L}}_+ = \frac{1}{2}\left[r\frac{d}{dr} - \frac{1}{2}r^2 + \left(2n + \beta + \frac{3}{2}\right)\right], \tag{92}$$

satisfying the equation

$$\hat{\mathcal{L}}_+|n\rangle_\beta = \ell_+|n+1\rangle_\beta = \sqrt{(n+1)(\beta+n+1)}|n+1\rangle_\beta. \tag{93}$$

We now study the algebra associated to the operators $\hat{\mathcal{L}}_+$ and $\hat{\mathcal{L}}_-$. On the basis of results (89) and (93) we can calculate the commutator $[\hat{\mathcal{L}}_-, \hat{\mathcal{L}}_+]$:

$$[\hat{\mathcal{L}}_-, \hat{\mathcal{L}}_+|n\rangle_\beta = 2\ell_0|n\rangle_\beta, \tag{94}$$

where we have introduced the eigenvalue

$$\ell_0 = \left(n + \frac{\beta+1}{2}\right). \tag{95}$$

We can thus define the operator

$$\hat{\mathcal{L}}_0 = \left(\hat{n} + \frac{\beta+1}{2}\right). \tag{96}$$

The operators $\hat{\mathcal{L}}_\pm$ and $\hat{\mathcal{L}}_0$ thus satisfy the commutation relations

$$[\hat{\mathcal{L}}_-, \hat{\mathcal{L}}_+] = 2\hat{\mathcal{L}}_0, \qquad [\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_-] = -\hat{\mathcal{L}}_-, \qquad [\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_+] = \hat{\mathcal{L}}_+ \tag{97}$$

which correspond to the SU(1, 1) group for the pseudoharmonic oscillator. The Casimir operator can be also expressed as

$$\begin{aligned}\hat{C}|n\rangle_\beta = [\hat{\mathcal{L}}_0(\hat{\mathcal{L}}_0 - 1) - \hat{\mathcal{L}}_+\hat{\mathcal{L}}_-]|n\rangle_\beta &= [\hat{\mathcal{L}}_0(\hat{\mathcal{L}}_0 + 1) - \hat{\mathcal{L}}_-\hat{\mathcal{L}}_+]|n\rangle_\beta \\ &= J(J-1)|n\rangle_\beta \end{aligned} \tag{98}$$

with

$$J = \frac{\beta+1}{2}. \tag{99}$$

Finally, we should notice that the Hamiltonian acquires the simple form

$$\hat{H} = \hat{\mathcal{L}}_0 - \frac{r_0^2}{4}. \tag{100}$$

For further calculations one can obtain the following expressions in terms of the creation and annihilation operators $\hat{\mathcal{L}}_\pm$ and $\hat{\mathcal{L}}_0$ as

$$r^2 = 2[2\hat{\mathcal{L}}_0 - (\hat{\mathcal{L}}_+ + \hat{\mathcal{L}}_-)] \tag{101}$$

and

$$r\,\frac{d}{dr} = (\hat{\mathcal{L}}_+ - \hat{\mathcal{L}}_-) - \frac{1}{2}. \tag{102}$$

The matrix elements of these two functions can be analytically obtained in terms of Eqs. (89) and (93) as

$$\begin{aligned}\langle m|r^2|n\rangle = 2\big[&(2n+\beta+1)\delta_{m,n} - \sqrt{(n+1)(n+\beta+1)}\delta_{m,n+1} \\ &- \sqrt{n(n+\beta)}\delta_{m,n-1}\big]\end{aligned} \tag{103}$$

and

$$\left\langle m \left| \frac{d}{dr} \right| n \right\rangle = \sqrt{(n+1)(n+\beta+1)}\delta_{m,n+1} - \sqrt{n(n+\beta)}\delta_{m,n-1} - \frac{1}{2}\delta_{m,n}. \tag{104}$$

2.4. Infinitely Deep Square-Well Potential

The Hamiltonian for a single particle moving in a one-dimensional infinitely deep square-well potential is

$$H = \frac{p^2}{2\mu} + V(x),$$
$$V(x) = \begin{pmatrix} 0 & 0 \leq x \leq L \\ \infty & \text{otherwise}, \end{pmatrix} \tag{105}$$

whose wavefunctions become

$$|n\rangle \equiv \psi_n(x) = \sqrt{\frac{1}{\pi}}\ \sin(nx), \qquad E_n = \frac{\hbar^2}{2\mu}n^2, \quad n = 1, 2, 3 \ldots. \tag{106}$$

For convenience we define the "number" operator $\hat{n}$

$$\hat{n}|n\rangle = n|n\rangle. \tag{107}$$

We now address how to find the creation and annihilation operators $\hat{S}_{\pm}$ from the wave functions (106)

$$\hat{\mathcal{S}}_{\pm}|n\rangle = s_{\pm}|n \pm 1\rangle. \tag{108}$$

From the wavefunctions (106) we have

$$\frac{d}{dx}\psi_n(x) = n\sqrt{\frac{1}{\pi}}\ \cos(nx). \tag{109}$$

We thus express $\hat{\mathcal{S}}_{\pm}$ by x, d/dx and $\hat{n}$ as

$$\hat{\mathcal{S}}_{-} = \left[(\cos x)\hat{n} - (\sin x)\,\frac{d}{dx}\right]\frac{\hat{n}-1}{\hat{n}}, \qquad \hat{\mathcal{S}}_{+} = (\cos x)\hat{n} + (\sin x)\,\frac{d}{dx}, \tag{110}$$

which implies that

$$\hat{\mathcal{S}}_{-}|n\rangle = s_{-}|n-1\rangle = (n-1)|n-1\rangle, \qquad \hat{\mathcal{S}}_{+}|n\rangle = s_{+}|n+1\rangle = n|n+1\rangle. \tag{111}$$

The commutator $[\hat{\mathcal{S}}_{-}, \hat{\mathcal{S}}_{+}]$ can be calculated on the basis $|n\rangle$

$$[\hat{\mathcal{S}}_{-}, \hat{\mathcal{S}}_{+}]|n\rangle = (2n-1)|n\rangle = 2\hat{\mathcal{S}}_0|n\rangle, \tag{112}$$

which implies that

$$\hat{\mathcal{S}}_0 = \hat{n} - 1/2. \tag{113}$$

At least, in the spaces spanned by $|n\rangle$ the operators $\hat{\mathcal{S}}_+$ and $\hat{\mathcal{S}}_0$ satisfy the commutation relations of an SU(1, 1) algebra, which is isomorphic to the SO(2, 1) algebra

$$[\hat{\mathcal{S}}_-, \hat{\mathcal{S}}_+] = 2\hat{\mathcal{S}}_0, \qquad [\hat{\mathcal{S}}_0, \hat{\mathcal{S}}_\pm] = \pm\hat{\mathcal{S}}_\pm, \tag{114}$$

which is the dynamical group for the infinitely square-well potential.

3. HARMONIC LIMITS OF THE SU(2) ALGEBRA

3.1. The Case of the Morse Potential

In this section we turn our attention to the harmonic limit in which the Morse potential approaches a harmonic oscillator potential. In this limit $\beta \to 0$ and $V_0 \to \infty$, but keeping the product $k = 2\beta^2 V_0$ finite, so that the expansion of the exponential functions in (3), leads to the harmonic limit

$$\lim_{V_0 \to \infty} V_{\mathrm{Morse}} = \frac{1}{2} k x^2. \tag{115}$$

We now proceed to analyze the contraction of the SU(2) algebra

$$G_{\mathrm{SU(2)}} = \{\hat{K}_+, \hat{K}_-, \hat{K}_0\} \tag{116}$$

for this limit. We first note that according to the relation $2s = \nu - 2n - 1$, we have

$$\lim_{\nu \to \infty} \frac{2s}{\nu} = \lim_{\nu \to \infty} \sqrt{\frac{s-1}{s}} = \lim_{\nu \to \infty} \sqrt{\frac{s+1}{s}} = 1. \tag{117}$$

If we now expand the exponential function of the variable y keeping in mind that in the harmonic limit $\beta \to 0$, we find the approximation

$$y \simeq \nu(1 - \beta x); \qquad \frac{1}{y} \simeq \frac{1}{\nu}(1 + \beta x), \tag{118}$$

which can be used to obtain the corresponding approximation for the derivative

$$\frac{d}{dy} = -\frac{1}{\beta}\frac{1}{y}\frac{d}{dx}, \tag{119}$$

whose harmonic limit turns out to be

$$\lim_{\nu \to \infty} \frac{d}{dy} = \lim_{\nu \to \infty} \left[-\frac{1}{\beta}\frac{1}{\nu}(1 + \beta x)\frac{d}{dx} \right] = -\frac{1}{\beta\nu}\frac{d}{dx}. \tag{120}$$

We are now ready to study the harmonic limit of the operators (116), but before doing so it is convenient to introduce the renormalization

$$b^\dagger = \frac{\hat{K}_+}{\sqrt{\nu}}; \qquad b = \frac{\hat{K}_-}{\sqrt{\nu}}; \qquad b_0 = \frac{-2\hat{K}_0}{\sqrt{\nu}}, \tag{121}$$

which, when considered in (10) and (19), leads to

$$\lim_{\nu\to\infty} b^\dagger = \frac{\sqrt{\nu}\beta}{2}x - \frac{1}{\beta\sqrt{\nu}}\frac{d}{dx} = \sqrt{\frac{\mu\omega}{2\hbar}}x - \sqrt{\frac{\hbar}{2\mu\omega}}\frac{d}{dx} = a^\dagger; \tag{122a}$$

$$\lim_{\nu\to\infty} b = \sqrt{\frac{\mu\omega}{2\hbar}}x + \sqrt{\frac{\hbar}{2\mu\omega}}\frac{d}{dx} = a; \tag{122b}$$

$$\lim_{\nu\to\infty} b_0 = 1, \tag{122c}$$

with ω given by (29). The operators a^+ and a satisfy the bosonic commutation relation

$$[a, a^\dagger] = 1; \qquad [a, a] = [a^\dagger, a^\dagger] = 0, \tag{123}$$

as expected. Thus, in the harmonic limit the SU(2) algebra contracts to the Weyl algebra, i.e.,

$$\lim_{\nu\to\infty} G_{\mathrm{SU(2)}} = \{a^\dagger, a, 1\}. \tag{124}$$

Finally, in terms of the operators (121), the Morse wave functions take the simple form

$$|n\rangle_\nu = \sqrt{\frac{\nu^n(\nu-n-1)!}{n!(\nu-1)!}}(b^\dagger)^n|0\rangle_\nu, \tag{125}$$

whose harmonic limit is given by

$$\lim_{\nu\to\infty} |n\rangle_\nu = \frac{1}{\sqrt{n!}}(a^\dagger)^n\phi_0(y), \tag{126}$$

where $\phi_0(y)$ is the ground state for the harmonic oscillator.

Before finishing this section, it is interesting to note that the operators $\hat{b}$ and $\hat{b}^\dagger$ can be explicitly expressed in terms of the physical coordinate x and its corresponding momentum $\hat{p}$:

$$\hat{b}^\dagger = \left[\frac{e^{\beta x}}{\nu}\left(-\frac{i\hat{p}}{\beta\hbar} + s\right)(2s-1) - \frac{\nu}{2}\right]\sqrt{\frac{s-1}{\nu s}}, \tag{127a}$$

$$\hat{b} = \left[\frac{e^{\beta x}}{\nu}\left(\frac{i\hat{p}}{\beta\hbar} + s\right)(2s+1) - \frac{\nu}{2}\right]\sqrt{\frac{s+1}{\nu s}}, \tag{127b}$$

3.2. The Case of the MPT Potential

We now analyze the harmonic limit of the MPT potential, which is obtained when $\alpha \to 0$ and $D \to \infty$, but keeping the product $k = 2\alpha^2 D$ finite, so that the expansion of the exponential functions in (36), leads to

$$\lim_{D\to\infty} V_{\mathrm{MPT}} = \frac{1}{2}kx^2. \tag{128}$$

In the algebraic scheme this limit must be applied to the SU(2) generators, which are convenient to be renormalized in the following form

$$\hat{b}^\dagger = \frac{\hat{P}_+}{\sqrt{\nu}} \qquad \hat{b} = \frac{\hat{P}_-}{\sqrt{\nu}} \qquad \hat{b}_0 = \frac{-2\hat{P}_0}{\nu}. \tag{129}$$

We first note that according to the relation $2\epsilon = \nu - 2n - 1$, we have

$$\lim_{\nu\to\infty} \sqrt{\frac{\epsilon + 1}{\epsilon}} = \lim_{\nu\to\infty} \sqrt{\frac{\epsilon - 1}{\epsilon}} = 1. \tag{130}$$

On the other hand, we can make the approximation $\cosh(\alpha x) \simeq 1$ and $\sinh(\alpha x) \simeq \alpha x$ in the harmonic limit $\alpha \to 0$. These results, together with (52) and (57), lead to

$$\lim_{\nu\to\infty} \hat{b}^\dagger = \left(-\frac{1}{\sqrt{\nu}\alpha}\frac{d}{dx} + \frac{\sqrt{\nu}\alpha}{2}x\right) = \sqrt{\frac{\mu\omega}{2\hbar}}x - \sqrt{\frac{\hbar}{2\mu\omega}}\frac{d}{dx} = \hat{a}^\dagger; \tag{131a}$$

$$\lim_{\nu\to\infty} \hat{b} = \left(\frac{1}{\sqrt{\nu}\alpha}\frac{d}{dx} + \frac{\sqrt{\nu}\alpha}{2}x\right) = \sqrt{\frac{\mu\omega}{2\hbar}}x + \sqrt{\frac{\hbar}{2\mu\omega}}\frac{d}{dx} = \hat{a}; \tag{131b}$$

$$\lim_{\nu\to\infty} \hat{b}_0 = 1, \tag{131c}$$

with

$$\omega = \frac{\alpha^2\hbar}{2\mu}\nu \simeq \sqrt{\frac{2D\alpha^2}{\mu}}, \tag{132}$$

where we have made use of the relation $\nu/2 - k \simeq \sqrt{\frac{2\mu D}{\alpha^2\,\hbar^2}}$ in the harmonic limit $D \to \infty$. The operators $\hat{a}^\dagger$ and $\hat{a}$ satisfy the bosonic commutation relations

$$[\hat{a}, \hat{a}^\dagger] = 1; \qquad [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0, \tag{133}$$

as expected. Therefore the SU(2) algebra is contracted to the Weyl algebra in the harmonic limit

$$\lim_{\nu\to\infty} G_{\mathrm{SU(2)}} = \lim_{\nu\to\infty}\{\hat{b}^\dagger, \hat{b}, \hat{b}_0\} = \{\hat{a}^\dagger, \hat{a}, 1\}. \tag{134}$$

Finally, in terms of the operators (129), the MPT wave functions can be simply expressed as

$$|n\rangle_q = \sqrt{\frac{\nu^n(\nu - n - 1)!}{n!(\nu - 1)!}}(\hat{b}^\dagger)^n|0\rangle_q, \tag{135}$$

whose harmonic limit is given by

$$\lim_{\nu\to\infty}|n\rangle_q = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\phi_0(u), \tag{136}$$

where $\phi_0(u)$ is the ground state of the harmonic oscillator.

4. CONCLUSIONS

In this paper, we established the raising and lowering operators for important molecular potentials, such as the Morse potential, the MPT potential, infinitely deep square-well potential, and the pseudoharmonic potential. We derived the realizations only in terms of the physical variable without introducing an auxiliary variable. It is shown that the SU(2) group was the appropriate dynamical symmetry for the bound states of the Morse and MPT potentials, but the SU(1, 1) group for the infinitely deep square-well potential and the pseudoharmonic potential. We used the SU(2) algebra to express the Morse and MPT wave functions in terms of the action of the creation operator $\hat{K}_+$ and $\hat{P}_+$ on the ground state. The matrix elements of the different related functions were analytically obtained in terms of the ladder operators. This method can be generalized to other functions and represents a simple and elegant approach to obtain these matrix elements in comparison with the traditional techniques in configuration space. The harmonic limits were also analyzed, showing that the SU(2) algebra for the Morse and MPT potentials contracts to the appropriate Weyl algebra in this limit.

ACKNOWLEDGMENTS

S. H. Dong thanks Prof. A. Frank for the invitation to UNAM. This work is supported by CONACyT, Mexico, under project 32397-E.

REFERENCES

Alhassid, Y., Gürsey, F., and Iachello, F. (1983). *Annals of Physics* **148**, 346.
Arima, A. and Iachello, F. (1974). *Annals of Physics* **99**, 253.
Ballhausen, C. J. (1988). *Chemical Physics Letters* **146**, 449.
Benedict, M. G. and Molnar, B. (1999). *Physical Review A* **60**, R1737.
Berceanu, S. and Gheorghe, G. (1987). *Journal of Mathematical Physics* **28**, 2899.
Berrondo, M. and Palma, A. (1980). *Journal of Physics A: Mathematical and General* **13**, 773.

Bessis, N. and Bessis, G. (1994). *Physical Review A* **50**, 4506.
Büyükkilic, F., Demirhan, D., and ÖZeren, S. F. (1992). *Chemical Physics Letters* **194**, 9.
Child, M. S. and Halonen, L. (1984). *Advances in Chemical Physics* **62**, 1.
Cooper, I. L. (1993). *Journal of Physics A: Mathematical and General* **26**, 1601.
Englefield, M. J. and Quesne, C. (1991). *Journal of Physics A: Mathematical and Genernal* **24**, 3557.
Flügge, S. (1971). Practical *Quantum Mechanics*, Springer-Verlag, Berlin, p. 94.
Frank, A. and Van Isacker, P. (1994). *Algebraic Methods in Molecular and Nuclear Structure Physics*, Wiley, New York.
Frank, A. and Wolf, K. B. (1984). *Physical Review Letters* **52**, 1737.
Gerry, C. C. (1986). *Physical Review A* **33**, 2207.
Goldman, I. I., Krivchenkov, V. D., Kogan, V. A., and Galitskii, V. M. (1960). *Problems in Quantum Mechanics*, London, p. 8.
Gradshteyn, I. S. and Ryzhik, I. M. (1994). *Tables of Integrals, Series, and Products*, 5th edn., Academic Press, New York.
Iachello, F. and Levine, R. D. (1995). *Algebraic Theory of Molecules*, Oxford University Press, Oxford.
Iachello, F. and Oss, S. (1993). *Journal of Chemical Physics* **99**, 7337.
Infeld, L. and Hull, T. E. (1951). *Reviews of Morden Physics*. **23**, 21.
Jensen, P. (2000). *Molecular Physics* **98**, 1253.
Kais, S. and Levine, R. D. (1990). *Physical Review A* **41**, 2301.
Kondo, A. E. and Truax, R. (1988). *Journal of Mathematical Physics* **29**, 1396.
Landau, L. D. and Lifshitz, E. M. (1977). *Quantum Mechanics, Non-Relativistic Theory*, 3rd ed., Pergamon, Oxford, UK, p. 72.
Morse, P. M. (1929). *Physical Review* **34**, 57.
Nieto, M. M. and Simmons, L. M., Jr. (1979). *Physical Review A* **19**, 435.
Popov, D. (2001). *Journal of Physics A: Mathematical and General* **34**, 5283.
Pöschl, G. and Teller, E. (1933). *Zeitschrift fur Physik* **83**, 143.
Sage, M. L. (1984). *Chemical Physics* **87**, 431.
Talman, J. D. (1968). *Special Functions: A group theoretic approach, Mathematical Physics Monograph Series*, New York, p. 182.
Wang, Z. X. and Guo, D. R. (1989). *Special Functions*, World Scientific, Singapore, p. 276.
Wu, J. and Alhassid, Y. (1990). *Journal of Mathematical Physics* **31**, 557.